Much Ado About Theta

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Introduces and explores the idea of establishing the complexity of a problem, as opposed to the complexity of an algorithm. This powerful idea is crucial to our quest for finding optimal data structures with which to implement our algorithms.

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Complexity of Problems

We have discussed the Θ notation as applied to bounding the growth-rate of the time required for an algorithm. That's a powerful tool, but the real strength of the Ω and Θ notations lies in using them to classify problems.

What would it mean to say that a problem P has a lower bound $f(n)$ on its complexity? It would mean that we can prove that **every possible algorithm** that solves P is in $\Omega(f(n))$.

How can we do that? We would have to prove the statement not only for all known algorithms that solve P, but also all algorithms that might be discovered in the future that solve this problem.

We can do this by making some simple assumptions about the computer architecture - basically that we are only considering sequential (non-parallel) machines, with constant-time arithmetic operations and a random-access memory.

This rules out possible breakthroughs such as effective quantum computing, hyperspace or time-travel.

Within these constraints, we can see immediately that any problem that requires reading n input values must be in $\Omega(n)$. This is kind of trivial but it is often the best we can do. It's worth pointing out that sometimes we ignore the input phase of an algorithm - the best example is binary search, which we always describe as being in $O(\log n)$. Obviously this is only true if we ignore the time required to input (and sort) the set of n values.

Sometimes we **can** do better than the trivial lower bound. For example, suppose a certain problem requires multiplying all pairs of values in a set of size n. There are about $\frac{n^2}{2}$ such pairs so any sequential algorithm that computes the necessary products must be in $\Omega(n^2)$.

Knowing the classification of a problem can help us in our quest to find an optimal algorithm for the problem.

And lest we forget, that **is** the ultimate goal of our study of data structures.

For example suppose we can show that a particular problem is in $\Omega(n^2)$ but the best algorithm we have is in $O(n^3)$. There are two avenues for exploration and research. If we can show that the problem is in $\Omega(n^3)$ then we know the problem is in $\Theta(n^3)$ and the known algorithm has optimal complexity. On the other hand, if we find an $O(n^2)$ algorithm for this problem then we can say the **problem** is in $\Theta(n^2)$ - every algorithm for this problem has running time that grows at least as fast as n^2 grows (because of the $\Omega(n^2)$ lower bound), and we have found an algorithm that grows exactly that fast. So n^2 is both a lower and upper bound on the complexity of solving this problem.

There is a famous and deeply studied problem that must be mentioned here: MATRIX MULTIPLICATION. Given two $n \times n$ matrices, we wish to compute their product. Since we have to input $2n^2$ values this problem is clearly in $\Omega(n^2)$. The naïve matrix multiplication algorithm runs in $O(n^3)$ time. For decades people have been finding faster and faster algorithms for this problem, but its true complexity has yet to be determined - nobody has found an algorithm that runs in $O(n^2)$ time.

The first major breakthrough on this problem was made by **Volker Strassen** in 1969.

EXAMPLE: Let's look at a simple example of determining the classification of a problem. The problem we will look at is evaluating a polynomial

$$f(x) = a_t * x^n + a_{t-1} * x^{n-1} + \cdots + a_1 * x + a_0$$

First we can observe that any algorithm that solves this must at the very least read or otherwise receive the values of x and all n of the a_i coefficients. Thus we can easily see that every algorithm for this problem must be in $\Omega(n)$.

Consider the simple algorithm I will call **BFI_Poly**:

```
BFI_Poly(x,a[n] ... a[0]):

value = a[0]
for i = 1 .. n:
    power = 1
    for j = 1 .. i:
        power *= x
    value += a[i]*power
return value
```

BFI_Poly() runs in time $O(n^2)$.

You should verify this.

So we have a problem with a lower bound of $\Omega(n)$, and an algorithm that is in $O(n^2)$ ... can we either increase the lower bound, or decrease the upper bound?

It turns out that for this problem we can decrease the upper bound by using a better algorithm - namely, Horner's Rule:

If you are not familiar with Horner's algorithm you should make sure you understand it. It is a perfect example of computational optimization.

```
Horners_Poly(x,a[n] ... a[0]):

value = a[n]
for i = n-1 .. 0:
    value = value*x + a[i]
return value
```

You should be able to verify that **Horners_Poly** runs in $O(n)$ time.

As an exercise, can you find an easy way to modify **BFI_Poly** so that it also runs in $O(n)$ time?

Now we are in clover - the upper bound on our algorithm exactly matches the lower bound on the problem. We can now say that the problem is in $\Theta(n)$. This really is very good news - it means we have found an algorithm for this problem that cannot be beat!

Well ... sort of.

It means our algorithm belongs to the lowest possible complexity class for solving this problem. There may be another algorithm with the same complexity and a lower value of c, the constant multiplier. This is what we see when we compare Mergesort and Quicksort: they have the same $O(n \log n)$ complexity, but Quicksort is faster in general because it has a lower constant multiplier.

Yes, I know that Quicksort has worst-case $O(n^2)$ complexity the way it is normally implemented. It is actually possible to modify Quicksort so that you can guarantee $O(n \log n)$ performance but hardly anyone bothers because the pathological situations that give rise to the $O(n^2)$ performance are very rare.

Comparison-Based Sorting

THE INFORMATION IN THIS SECTION may be considered as optional ... but I strongly recommend learning it!

The study of Θ classification has led to an incredibly important result in complexity theory with direct implications for algorithm and data structure design: comparison-based sorting of a set is in $\Theta(n \log n)$ where n is the size of the set. In other words, there **cannot** be any sorting algorithm based on comparing elements of the set to each other that is guaranteed to run in less than $\Omega(n \log n)$ time.

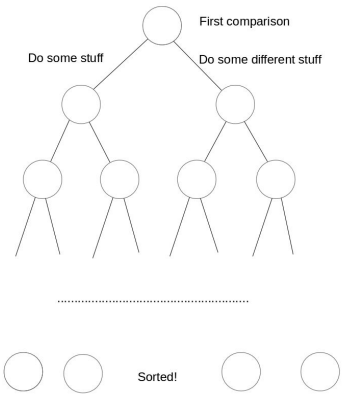
Despite what you may read on the internet about miraculous sorting algorithms that always run in $O(n)$ time, they don't exist.

A word about comparison-based sorting: most of the sorting algorithms we encounter are in this category. Bubble-sort for example, (which we all know we should never use in most circumstances because it runs in $O(n^2)$ time) is based on repeatedly comparing two consecutive values in the array and swapping them if required. Merge-sort boils down to a sequence of ever-larger merges, each of which consists of repeated comparisons between elements of the set. Quick Sort uses comparisons between values to partition the set into "small values" and "large values", then sorts the two subsets recursively. Each of these can be expressed at the most abstract level as:

```
while (not sorted):
    compare two elements of the set
    based on the result of the comparison, do some stuff
```

So the question is: if we have a sorting algorithm that fits this pattern, can we put a lower bound on the number of comparisons we must do? It turns out that we can!

We can visualize the execution of such an algorithm as a binary tree (note that this does not mean that the algorithm involves building a tree ... in this analysis the tree is a representational device for the execution of the algorithm). The root of the tree represents the first comparison. There are two possible outcomes, each leading to another comparison ... and each of those leads to two more, etc., until the set is sorted.



This tree has to include every possible sequence of comparisons that the algorithm might use to complete the sorting operation (ie. every possible execution trace of the algorithm). Every possible initial permutation of the set of n values will follow a different sequence of comparisons to become sorted, so each leaf of this tree represents the termination of the algorithm for a different initial permutation. Since

a set of n values has $n!$ permutations, the execution tree must have $n!$ leaves.

Now we are almost done. We can use the number of levels of the tree to put a lower bound on the running time of the algorithm. (For example, if the tree has 12 levels then there is some leaf that is only reached after 11 comparisons.) If we actually built this tree for bubble-sort we would see that it has about $c * n^2$ levels for some constant c, and if we built the execution trees for merge-sort we would see that the tree has about $d * n * \log n$ levels for some constant d.

But can we say anything about the minimum height of a binary tree with $n!$ leaves? If we think about this for a moment, we can see that if a binary tree has X leaves at the bottom level, then the level above this has $\leq \frac{X}{2}$ vertices, the one above that has $\leq \frac{X}{4}$ vertices, and so on up to the root. In other words the number of levels is at least $\log_2 X$.

So the execution tree for any possible comparison-based sorting algorithm must have at least $\log_2(n!)$ levels. This means that there is *at least one* permutation that requires at least $\log_2 n$ comparisons.

Because of the way logs work we get

$$\begin{aligned}
\log_2(n!) &= \log_2(1 * 2 * \cdots * \frac{n}{2} * (\frac{n}{2}+1) * \cdots * n) \\
&= \log_2(1) + \log_2(2) + \cdots + \log_2(\frac{n}{2}) + \log_2(\frac{n}{2}+1) + \cdots + \log_2(n) \\
&\geq \log_2(\frac{n}{2}) + \log_2(\frac{n}{2}+1) + \cdots + \log_2(n) \\
&\geq \log_2(\frac{n}{2}) + \log_2(\frac{n}{2}) + \cdots + \log_2(\frac{n}{2}) \\
&\geq \frac{n}{2}\log_2(\frac{n}{2}) \\
&= \frac{n}{2}(\log_2(n) - \log_2(2)) \\
&= \frac{n}{2}(\log_2(n) - 1)
\end{aligned}$$

which we now know means that we can write

$$\log_2(n!) \in \Omega(n \log n)$$

And there it is! The execution tree for any comparison-based sort algorithm must have at least $c * (n \log n)$ levels for some constant c, and so every comparison-based sorting algorithm that can successfully sort all possible initial permutations is in $\Omega(n \log n)$.

End of the sorting story? Not quite. If we place restrictions on the initial permutation - so that not all initial permutations are possible - then we may be able to get a lower complexity because the execution tree does not need as many leaves. Also, there do exist sorting algorithms that are not comparison-based - under some circumstances these can run faster than $O(n \log n)$ time. But for general purpose, no-restrictions sorting, the result holds.

Stories never end!

Congratulations if you made it to the end of this section . . . it's a result that crops up over and over in the study of algorithmic efficiency. Knowing why it is true puts you on very solid ground.